REPRESENTATIONS OF THE YOKONUMA–TEMPERLEY–LIEB ALGEBRA

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Abstract. We determine the representations of the Yokonuma–Temperley–Lieb algebra, which is defined as a quotient of the Yokonuma–Hecke algebra by generalising the construction of the classical Temperley–Lieb algebra.

1. Introduction

The Yokonuma–Hecke algebra, denoted by $\mathrm{Y}_{d,n}(u)$, was originally introduced by Yokonuma [Yo] in the context of Chevalley groups as a generalisation of the Iwahori–Hecke algebra of type A. It has interesting topological interpretations in the context of framed knots and links, because it can also arise as a quotient of the framed braid group algebra. Juyumaya and Lambropoulou used $\mathrm{Y}_{d,n}(u)$ to define a knot invariant for framed knots [JuLa1, JuLa2]. They subsequently proved that this invariant can be extended to classical and singular knots [JuLa3, JuLa4].

Moreover, the Yokonuma–Hecke algebra $\mathrm{Y}_{d,n}(u)$ can be regarded as a deformation of the group algebra of the complex reflection group $G(d,1,n) \cong (\mathbb{Z}/d\mathbb{Z}) \wr S_n$, where S_n denotes the symmetric group on n letters. For $d=1$, the algebra $\mathrm{Y}_{1,n}(u)$ coincides with the Iwahori–Hecke algebra $\mathcal{H}_n(u)$ of type A.

Some information on the representation theory of $\mathrm{Y}_{d,n}(u)$ has been obtained by Thiem in the general context of unipotent Hecke algebras [Th]. In a recent paper with Poulain d'Andecy, we have given an explicit description of the irreducible representations of $\mathrm{Y}_{d,n}(u)$ in terms of d-partitions and standard d-tableaux [ChPdA].

Now, the classical Temperley–Lieb algebra can be defined as the quotient of the Iwahori–Hecke algebra of type A over a certain ideal. In [GoJuLa], Goundaroulis, Juyumaya and Lambropoulou generalised this construction to the case of $\mathrm{Y}_{d,n}(u)$, and introduced the Yokonuma–Temperley–Lieb algebra, denoted by $\mathrm{YTL}_{d,n}(u)$, with the aim of studying its topological properties. Our aim in this paper will be to study the representations of this new object. In fact, we will determine which representations of $\mathrm{Y}_{d,n}(u)$ pass to the quotient $\mathrm{YTL}_{d,n}(u)$.

Thanks to Tits's deformation theorem, we will transfer our problem to the group algebra case, and study the representations of the analoguous quotient of $G(d,1,n)$. We will then transform the problem to a study of the restriction of representations from $G(d,1,n)$ to the symmetric group S_n. As we will see in Section 4, this restriction is controlled by the Littlewood–Richardson coefficients, which we will introduce in the beginning of this paper. Applying the Littlewood–Richardson rule in our case will allow us to obtain a complete classification of the irreducible representations of $\mathrm{YTL}_{d,n}(u)$ (in the semisimple case). Knowing thus the dimensions of the irreducible representations of $\mathrm{YTL}_{d,n}(u)$, we can finally compute the dimension of $\mathrm{YTL}_{d,n}(u)$ in Section 5.

2. The Littlewood–Richardson coefficients

In the first section of this paper we will introduce all the combinatorial tools that we will need in order to study the representation theory of the Yokonuma–Temperley–Lieb algebra.

2.1. **Partitions.** A partition is a family of positive integers $\lambda = (\lambda_1, \ldots, \lambda_k)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$. We set $|\lambda| := \lambda_1 + \cdots + \lambda_k$, and we call $|\lambda|$ the *size* of λ.

Let $n \in \mathbb{N}$. If λ is a partition such that $|\lambda| = n$, we say that λ is a *partition of* n. We will denote by $\mathcal{P}(n)$ the set of all partitions of n. We also set $\mathcal{P} := \bigcup_{n \geq 0} \mathcal{P}(n)$, the set of all partitions (including the empty one).

We identify partitions with their *Young diagrams*: the Young diagram $Y(\lambda)$ of λ is a left-justified array of k rows such that the j-th row contains λ_j *nodes* for all $j = 1, \ldots, k$. We write $\theta = (x, y)$ for the node in row x and column y. A node $\theta \in \lambda$ is called *removable* if the set of nodes obtained from λ by removing θ is still a partition.

2.2. **Skew shapes.** If $\nu, \lambda \in \mathcal{P}$ are such that $Y(\lambda) \subseteq Y(\nu)$, we can define the *skew shape* ν/λ; we have $Y(\nu/\lambda) = Y(\nu) \setminus Y(\lambda)$ and $|\nu/\lambda| = |\nu| - |\lambda|$. We will denote by $\mathcal{S}$ the set of all skew shapes.

Let $\nu/\lambda \in \mathcal{S}$ and let μ be a partition such that $|\mu| = |\nu/\lambda|$. A *skew semistandard tableau* T of shape ν/λ and weight μ is a way of filling the boxes of $Y(\nu/\lambda)$ with entries in $\{1, 2, \ldots, |\mu|\}$ such that:

- $T_{i,j} < T_{i+1,j}$ (the entries strictly increase down the columns);
- $T_{i,j} \leqslant T_{i,j+1}$ (the entries increase along the rows);
- μ_i corresponds to the number of entries equal to i.

We will denote by $\mathrm{SST}(\nu/\lambda)$ the set of all skew semistandard tableaux of shape ν/λ (with any possible weight).

Example 1. *Some skew semistandard tableaux of shape* $(4, 3, 2)/(2, 1)$ *and weight* $(3, 2, 1)$*:*

$$T_1 = \ytableaushort{\none\none11,\none12,23} \qquad T_2 = \ytableaushort{\none\none11,\none22,13} \qquad T_3 = \ytableaushort{\none\none12,\none12,13}$$

2.3. **Littlewood–Richardson tableaux.** For the definition of Littlewood–Richardson tableaux, we follow [Le]. Let ν/λ, $\nu'/\lambda' \in \mathcal{S}$. For $T \in \mathrm{SST}(\nu/\lambda)$ and $k, l \in \mathbb{N}$, we define T_k^l to be the number of entries l in row k of T. Two tableaux $T \in \mathrm{SST}(\nu/\lambda)$ and $T' \in \mathrm{SST}(\nu'/\lambda')$ are called *companion tableaux* if $T_k^l = T_l'^k$ for every $k, l \in \mathbb{N}$. In this case, T is called ν'/λ'*-dominant* (and T' is ν/λ-dominant). Note that if T is ν'/λ'-dominant, then the weight of T is equal to $\nu' - \lambda' := (\nu_1' - \lambda_1', \nu_2' - \lambda_2', \ldots)$.

We shall simply say that T is λ'*-dominant* if there exists a partition ν' such that T is ν'/λ'-dominant.

Definition 1. A tableau $T \in \mathrm{SST}(\nu/\lambda)$ is a *Littlewood–Richardson tableau* if T is $\emptyset$-dominant.

Example 2. *In Example* 1*, the tableaux* T_1 *and* T_2 *are Littlewood–Richardson, whereas* T_3 *is not.*

Lemma 1. *All entries in the first row of a Littlewood–Richardson tableau are* 1*.*

Proof. Let T be a skew semistandard tableaux of shape $\nu/\lambda \in \mathcal{S}$ and weight $\mu \in \mathcal{P}$. Then T is Littlewood–Richardson if and only if it is a companion tableau to a semistandard tableau T' of shape μ. If there exists an entry different from 1 in the first row of T, then there exist at least two entries equal to 1 in the first column of T'. This contradicts the fact that the entries of T' strictly increase down the columns. $\square$

2.4. **Littlewood–Richardson coefficients.** Let $\lambda, \mu, \nu \in \mathcal{P}$. We define the *Littlewood–Richardson coefficient* $c^{\nu}_{\lambda,\mu}$ to be the number of Littlewood–Richardson tableaux of shape ν/λ and weight μ.

Remark 1. If $\nu/\lambda \notin \mathcal{S}$ or $|\nu/\lambda| \neq |\mu|$, then $c^{\nu}_{\lambda,\mu} = 0$. Moreover, we have

$$c^{\nu}_{\lambda,\emptyset} = \begin{cases} 1, & \text{if } \nu = \lambda \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad c^{\nu}_{\nu,\mu} = \begin{cases} 1, & \text{if } \mu = \emptyset \\ 0, & \text{otherwise.} \end{cases}$$

The Littlewood–Richardson coefficients arise when decomposing a product of two Schur functions as a linear combination of other Schur functions: for $\lambda \in \mathcal{P}$, we denote by s_λ the corresponding Schur function. We then have

$$\mathrm{s}_\lambda \mathrm{s}_\mu = \sum_{\nu \in \mathcal{P}(n)} c^{\nu}_{\lambda,\mu} \mathrm{s}_\nu \tag{2.1}$$

where $n = |\lambda| + |\mu|$. Equation (2.1) is known as the "Littlewood–Richardson rule". It implies, among other things, the commutativity and associativity of Littlewood–Richardson coefficients (see also [Mac], or [DaKo] for an alternative proof), that is,

$$c^{\nu}_{\lambda,\mu} = c^{\nu}_{\mu,\lambda} \tag{2.2}$$

and

$$\sum_{\sigma} c^{\sigma}_{\lambda,\mu} c^{\pi}_{\sigma,\nu} = \sum_{\tau} c^{\tau}_{\mu,\nu} c^{\pi}_{\lambda,\tau}. \tag{2.3}$$

The following property of the Littlewood–Richardson coefficients, that we prove here, is crucial for the results in Section 4.

Lemma 2. *Let $n \in \mathbb{N}$, and let $\lambda, \mu \in \mathcal{P}$ with $|\lambda| + |\mu| = n$. Set $\alpha := \lambda_1 + \mu_1$. Then*

(1) for all $\nu \in \mathcal{P}(n)$ with $\nu_1 > \alpha$, we have $c^{\nu}_{\lambda,\mu} = 0$;

(2) there exists $\nu \in \mathcal{P}(n)$ such that $\nu_1 = \alpha$ and $c^{\nu}_{\lambda,\mu} > 0$.

Proof. (1) If $\nu/\lambda \notin \mathcal{S}$, then $c^{\nu}_{\lambda,\mu} = 0$. If $\nu/\lambda \in \mathcal{S}$, then the first row of $Y(\nu/\lambda)$ has $\nu_1 - \lambda_1$ boxes. We have $\nu_1 - \lambda_1 > \alpha - \lambda_1 = \mu_1$. Thus, if T is a skew semistandard tableau of shape ν/λ and weight μ, then the first row of T must contain entries greater than 1. By Lemma 1, T cannot be Littlewood–Richardson. Hence, there exist no Littlewood–Richardson tableaux of shape ν/λ and weight μ, that is, $c^{\nu}_{\lambda,\mu} = 0$.
(2) Let ν be the partition of n defined by $\nu_i := \lambda_i + \mu_i$ for all $i \geq 1$. Then $\nu_1 = \alpha$, $\nu/\lambda \in \mathcal{S}$ and the i-th row of $Y(\nu/\lambda)$ has μ_i boxes. Let T be the skew semistandard tableau of shape ν/λ and weight μ obtained by filling every box of the i-th row of $Y(\nu/\lambda)$ with the entry i. Then T is $\mu/\emptyset$-dominant, and thus Littlewood–Richardson. We conclude that $c^{\nu}_{\lambda,\mu} > 0$. □

The next result shows what happens in the case where the weight of a Littlewood–Richardson tableau consists of just one row. This implies the so-called "Pieri's rule", which we will state in Section 4.

Lemma 3. *Let $n \in \mathbb{N}$, and let $\lambda, \mu \in \mathcal{P}$ with $|\lambda| + |\mu| = n$. Suppose that μ if of the form (l) for some $1 \leqslant l \leqslant n$. Let $\nu \in \mathcal{P}(n)$. We have $c^{\nu}_{\lambda,\mu} > 0$ if and only if the Young diagram of ν can be obtained from that of λ by adding l boxes, with no two in the same column.*

Proof. Let T be a skew semistandard tableaux of shape $\nu/\lambda \in \mathcal{S}$ and weight $\mu = (l)$. Then T has l boxes, and all entries in T are equal to 1. Thus, for T to be semistandard, each column ot T must have at most one box. We deduce that $Y(\nu)$ is obtained from $Y(\lambda)$ by adding l boxes, with no two in the same column. Note that, in this case, T is always Littlewood–Richardson. □

2.5. **Multipartitions.** Let $d \in \mathbb{N}$. A *d-partition* λ, or a *Young d-diagram*, of size n is a d-tuple of partitions such that the total number of nodes in the associated Young diagrams is equal to n. That is, we have $\lambda = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(d-1)})$ with $\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(d-1)}$ usual partitions such that $|\lambda^{(0)}| + |\lambda^{(1)}| + \cdots + |\lambda^{(d-1)}| = n$. We also say that λ is a *d-partition of n*. We denote by $\mathcal{P}(d,n)$ the set of d-partitions of n. We have $\mathcal{P}(1,n) = \mathcal{P}(n)$.

3. The Yokonuma–Temperley–Lieb algebra

3.1. **The Yokonuma–Hecke algebra** $\mathrm{Y}_{d,n}(u)$**.** Let d, $n \in \mathbb{N}$, $d \geq 1$. Let u be an indeterminate. The Yokonuma–Hecke algebra, denoted by $\mathrm{Y}_{d,n}(u)$, is a $\mathbb{C}[u, u^{-1}]$-associative algebra generated by the elements

$$g_1, \ldots, g_{n-1}, t_1, \ldots, t_n$$

subject to the following relations:

$$(3.1) \qquad \begin{array}{lrcll} (\mathrm{b}_1) & g_i g_j & = & g_j g_i & \text{for all } i, j = 1, \ldots, n-1 \text{ such that } |i-j| > 1, \\ (\mathrm{b}_2) & g_i g_{i+1} g_i & = & g_{i+1} g_i g_{i+1} & \text{for all } i = 1, \ldots, n-2, \\ (\mathrm{f}_1) & t_i t_j & = & t_j t_i & \text{for all } i, j = 1, \ldots, n, \\ (\mathrm{f}_2) & t_j g_i & = & g_i t_{s_i(j)} & \text{for all } i = 1, \ldots, n-1 \text{ and } j = 1, \ldots, n, \\ (\mathrm{f}_3) & t_j^d & = & 1 & \text{for all } j = 1, \ldots, n, \end{array}$$

where s_i is the transposition $(i, i+1)$, together with the quadratic relations:

$$(3.2) \qquad g_i^2 = 1 + (u-1)\, e_i + (u-1)\, e_i\, g_i \qquad \text{for all } i = 1, \ldots, n-1$$

where

$$(3.3) \qquad e_i := \frac{1}{d} \sum_{s=0}^{d-1} t_i^s t_{i+1}^{-s}.$$

It is easily verified that the elements e_i are idempotents in $\mathrm{Y}_{d,n}(u)$.

Let us denote by S_n the symmetric group on n letters. The group S_n is generated by the transpositions $s_1, \ldots, s_{n-1}$. If we specialise u to 1, the defining relations (3.1)–(3.2) become the defining relations for the complex reflection group $G(d, 1, n) \cong (\mathbb{Z}/d\mathbb{Z}) \wr S_n$. Thus, the algebra $\mathrm{Y}_{d,n}(u)$ is a deformation of the group algebra over $\mathbb{C}$ of $G(d, 1, n)$. Moreover, for $d = 1$, the Yokonuma–Hecke algebra $\mathrm{Y}_{1,n}(u)$ coincides with the Iwahori–Hecke algebra $\mathcal{H}_n(u)$ of type A. Hence, for $d = 1$ and u specialised to 1, we obtain the group algebra over $\mathbb{C}$ of the symmetric group S_n.

Furthermore, the relations (b_1), (b_2), (f_1) and (f_2) are defining relations for the classical framed braid group $\mathcal{F}_n \cong \mathbb{Z} \wr B_n$, where B_n is the classical braid group on n strands, with the t_j's being interpreted as the "elementary framings" (framing 1 on the jth strand). The relations $t_j^d = 1$ mean that the framing of each braid strand is regarded modulo d. Thus, the algebra $\mathrm{Y}_{d,n}(u)$ arises naturally as a quotient of the framed braid group algebra over the modular relations (f_3) and the quadratic relations (3.2). Moreover, relations (3.1) are defining relations for the modular framed braid group $\mathcal{F}_{d,n} \cong (\mathbb{Z}/d\mathbb{Z}) \wr B_n$, so the algebra $\mathrm{Y}_{d,n}(u)$ can be also seen as a quotient of the modular framed braid group algebra over the quadratic relations (3.2).

3.2. **Representations of** $\mathrm{Y}_{d,n}(u)$**.** In [ChPdA], we explicitly constructed the irreducible representations of $\mathrm{Y}_{d,n}(u)$ over $\mathbb{C}(u)$, and we showed that they are parametrised by the d-partitions of n. We denote by

$$\mathrm{Irr}(\mathrm{Y}_{d,n}(u)) = \{\rho^\lambda \mid \lambda \in \mathcal{P}(d, n)\}$$

the set of irreducible representations of the algebra $\mathbb{C}(u)\mathrm{Y}_{d,n}(u) := \mathbb{C}(u) \otimes_{\mathbb{C}[u,u^{-1}]} \mathrm{Y}_{d,n}(u)$. We also showed that the algebra $\mathbb{C}(u)\mathrm{Y}_{d,n}(u)$ is split semisimple, and that the specialisation $u \mapsto 1$ induces a bijection between $\mathrm{Irr}(\mathrm{Y}_{d,n}(u))$ and the set

$$\mathrm{Irr}(G(d, 1, n)) = \{E^\lambda \mid \lambda \in \mathcal{P}(d, n)\}$$

of irreducible representations of the group $G(d, 1, n)$ over $\mathbb{C}$. The last result can be also independently obtained with the use of Tits's deformation theorem (see, for example, [GePf, Theorem 7.4.6]).

Remark 2. The trivial representation is labelled by the d-partition $((n), \emptyset, \emptyset, \ldots, \emptyset)$, in every case.

3.3. **The Yokonuma–Temperley–Lieb algebra** $\mathrm{YTL}_{d,n}(u)$**.** Let $n \geq 3$. The Yokonuma–Temperley–Lieb algebra, denoted by $\mathrm{YTL}_{d,n}(u)$, is defined in [GoJuLa] as the quotient of the Yokonuma–Hecke algebra $\mathrm{Y}_{d,n}(u)$ by the two-sided ideal

$$I := \langle\, g_i g_{i+1} g_i + g_i g_{i+1} + g_{i+1} g_i + g_i + g_{i+1} + 1 \mid i = 1, 2, \ldots, n-2 \,\rangle.$$

For $d = 1$, the algebra $\mathrm{YTL}_{1,n}(u)$ coincides with the ordinary Temperley–Lieb algebra $\mathrm{TL}_n(u)$.

3.4. **Representations of** $\mathrm{YTL}_{d,n}(u)$**.** Since $\mathrm{YTL}_{d,n}(u)$ is a quotient of $\mathrm{Y}_{d,n}(u)$, by standard results in representation theory, we have that:

- The algebra $\mathbb{C}(u)\mathrm{YTL}_{d,n}(u) := \mathbb{C}(u) \otimes_{\mathbb{C}[u,u^{-1}]} \mathrm{YTL}_{d,n}(u)$ is split semisimple.
- The irreducible representations of $\mathbb{C}(u)\mathrm{YTL}_{d,n}(u)$ are in bijection with the irreducible representations ρ^λ of $\mathbb{C}(u)\mathrm{Y}_{d,n}(u)$ satisfying:

$$\rho^\lambda(g_i g_{i+1} g_i + g_i g_{i+1} + g_{i+1} g_i + g_i + g_{i+1} + 1) = 0 \text{ for all } i = 1, 2, \ldots, n-2. \tag{3.4}$$

Let us denote by $\mathcal{R}(d,n)$ the set of d-partitons λ of n for which (3.4) is satisfied. We denote by

$$\mathrm{Irr}(\mathrm{YTL}_{d,n}(u)) = \{\varrho^\lambda \mid \lambda \in \mathcal{R}(d,n)\}$$

the set of irreducible representations of the algebra $\mathbb{C}(u)\mathrm{YTL}_{d,n}(u)$. For every $\lambda \in \mathcal{R}(d,n)$, we have

$$\varrho^\lambda \circ \pi = \rho^\lambda,$$

where π is the natural surjective homomorphism from $\mathrm{Y}_{d,n}(u)$ onto $\mathrm{YTL}_{d,n}(u)$. The aim of this paper will be to determine the set $\mathcal{R}(d,n)$.

Proposition 1. *We have $\lambda \in \mathcal{R}(d,n)$ if and only if the trivial representation is not a direct summand of the restriction $\mathrm{Res}^{G(d,1,n)}_{S_3}(E^\lambda)$.*

Proof. Let us consider the quotient of the group algebra $\mathbb{C}G(d,1,n)$ by the two-sided ideal

$$J := \langle\, s_i s_{i+1} s_i + s_i s_{i+1} + s_{i+1} s_i + s_i + s_{i+1} + 1 \mid i = 1, 2, \ldots, n-2 \,\rangle.$$

The quotient algebra $A := \mathbb{C}G(d,1,n)/J$ is a split semisimple agebra over $\mathbb{C}$. For $u = 1$, the Yokonuma–Temperley–Lieb algebra specialises to A. By Tits's deformation theorem, the specialisation $u \mapsto 1$ yields a bijection between $\mathrm{Irr}(\mathrm{YTL}_{d,n}(u))$ and the set $\mathrm{Irr}(A)$ of irreducible representations of A. Thus, we can write:

$$\mathrm{Irr}(A) = \{\mathcal{E}^\lambda \mid \lambda \in \mathcal{R}(d,n)\}.$$

Now, following the same reasoning as for the Yokonuma–Temperley–Lieb algebra, $\mathcal{E}^\lambda \in \mathrm{Irr}(A)$ if and only if

$$E^\lambda(s_i s_{i+1} s_i + s_i s_{i+1} + s_{i+1} s_i + s_i + s_{i+1} + 1) = 0 \text{ for all } i = 1, 2, \ldots, n-2.$$

This equation is equivalent to

$$\mathrm{Res}^{G(d,1,n)}_{\langle s_i, s_{i+1}\rangle}(E^\lambda)(s_i s_{i+1} s_i + s_i s_{i+1} + s_{i+1} s_i + s_i + s_{i+1} + 1) = 0 \text{ for all } i = 1, 2, \ldots, n-2.$$

Now, the group $\langle s_i, s_{i+1}\rangle \cong S_3$ has three irreducible representations, corresponding to the partitions (3), $(2,1)$ and $(1,1,1)$. Among them, only the trivial representation, corresponding to the partition (3), does not evaluate to 0 on $s_i s_{i+1} s_i + s_i s_{i+1} + s_{i+1} s_i + s_i + s_{i+1} + 1$, whence the desired result. $\square$

Remark 3. An alternative proof for Proposition 1 can be obtained by looking directly at the representations of the Yokonuma–Hecke algebra and their restrictions to $\mathrm{Y}_{d,3}(u)$, with the use of the formulas obtained in [ChPdA].

Proposition 1 has transformed the problem of determination of the irreducible representations of $\mathrm{YTL}_{d,n}(u)$ to a problem of determination of the irreducible representations appearing in the restriction of a representation from $G(d,1,n)$ to S_3. For $d=1$, the restriction of an irreducible representation labelled by a partition λ corresponds to the removal of (removable) nodes from the Young diagram of λ. More specifically, if λ is a partition of n, then $\mathrm{Res}^{S_n}_{S_{n-1}}(E^\lambda)$ is the direct sum of all representations labelled by the partitions of $n-1$ whose Young diagrams are obtained from the Young diagram of λ by removing one node. As a consequence, $\mathrm{Res}^{S_n}_{S_k}(E^\lambda)$, where $k<n$, is a direct sum (with various multiplicities) of all representations labelled by the partitions of k whose Young diagrams are obtained from the Young diagram of λ by removing $n-k$ nodes. In particular, $\mathrm{Res}^{S_n}_{S_3}(E^\lambda)$ is a direct sum of all representations labelled by the partitions of 3 whose Young diagrams are obtained from the Young diagram of λ by removing $n-3$ nodes. Hence, the trivial representation is a direct summand of $\mathrm{Res}^{S_n}_{S_3}(E^\lambda)$ if and only if the Young diagram of λ has more than two columns. This implies the following corollary of Proposition 1:

Corollary 1. *We have $\lambda \in \mathcal{R}(d,n)$ if and only if all direct summands of $\mathrm{Res}^{G(d,1,n)}_{S_n}(E^\lambda)$ are labelled by partitions whose Young diagrams have at most two columns.*

This in turn yields the following well-known characterisation of the irreducible representations of the classical Temperley–Lieb algebra $\mathrm{TL}_n(u)$:

Corollary 2. *We have*

$$\mathcal{R}(1,n) = \{\lambda \in \mathcal{P}(n) \,|\, \lambda_1 \leqslant 2\}. \tag{3.5}$$

That is, $E^\lambda \in \mathrm{Irr}(\mathrm{TL}_n(u))$ if and only if the Young diagram of λ has at most two columns.

Unfortunately, for $d>1$, the restriction from $G(d,1,n)$ to S_n is far more complicated than in the symmetric group case. As we will see in the next section, the combinatorics of the restriction are governed by the Littlewood–Richardson coefficients, and it is in general difficult to judge which irreducible representations appear in $\mathrm{Res}^{G(d,1,n)}_{S_n}(E^\lambda)$. Nevertheless, the study of the Littlewood–Richardson coefficients in Section 4 will allow us to obtain the answer to our problem, that is, to determine $\mathcal{R}(d,n)$ for any $d \in \mathbb{N}$.

4. Restriction to S_n and the Littlewood–Richardson rule

Our aim in this section will be to study the restriction of representations from $G(d,1,n)$ to S_n. We will then use Corollary 1 to determine the set $\mathcal{R}(d,n)$, and thus the irreducible representations of the Yokonuma–Temperley–Lieb algebra $\mathrm{YTL}_{d,n}(u)$.

4.1. **Induction, restriction and the Littlewood–Richardson coefficients.** The Littlewood–Richardson coefficients control the induction to S_n from Young subgroups. Let $k,l \in \mathbb{N}$ be such that $k+l=n$. Let $\lambda \in \mathcal{P}(k)$ and $\mu \in \mathcal{P}(l)$. Then the Littlewood–Richardson rule yields (see, for example, [St, §3]):

$$\mathrm{Ind}^{S_n}_{S_k\times S_l}(E^\lambda \boxtimes E^\mu) = \sum_{\nu\in\mathcal{P}(n)} c^\nu_{\lambda,\mu} E^\nu. \tag{4.1}$$

More generally, if $\lambda^{(i)} \in \mathcal{P}(k_i)$ for $0 \leqslant i \leqslant d-1$ and $\sum_{i=0}^{d-1} k_i = n$, then

$$\mathrm{Ind}^{S_n}_{H}(E^{\lambda^{(0)}}\boxtimes E^{\lambda^{(1)}}\boxtimes\cdots\boxtimes E^{\lambda^{(d-1)}}) = \sum_{\nu^{(i)}\in\mathcal{P}(k_0+\cdots+k_i)} c^{\nu^{(1)}}_{\lambda^{(0)},\lambda^{(1)}} c^{\nu^{(2)}}_{\nu^{(1)},\lambda^{(2)}} \cdots c^{\nu^{(d-1)}}_{\nu^{(d-2)},\lambda^{(d-1)}} E^{\nu^{(d-1)}}, \tag{4.2}$$

where $H := \prod_{i=0}^{d-1} S_{k_i}$. Note that $\nu^{(d-1)} \in \mathcal{P}(n)$. Note also that, due to the commutativity and associativity of Littlewood–Richardson coefficients (Relations (2.2) and (2.3)), if $E^{\nu^{(d-1)}}$ appears with non-zero coefficient in (4.2), then $\nu^{(d-1)}/\lambda^{(i)} \in \mathcal{S}$ for all $i=0,1,\ldots,d-1$.

Now let G be any finite group. In order to obtain a complete set of irreducible representations for the wreath product $G \wr S_n$ (a problem originally solved by Specht [Sp]), one needs to consider representations induced from wreath analogues of Young subgroups. In the case where G is the cyclic group of order d, we have the following: Let $\lambda = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(d-1)}) \in \mathcal{P}(d,n)$, and let us consider the irreducible representation E^λ of $G(d,1,n) \cong (\mathbb{Z}/d\mathbb{Z}) \wr S_n \cong (\mathbb{Z}/d\mathbb{Z})^n \rtimes S_n$. For all $i = 0, 1, \ldots, d-1$, set $k_i := |\lambda^{(i)}|$. Then, by Specht's Theorem (see, for example, [St, Theorem 4.1]), we have

$$E^\lambda = \mathrm{Ind}_{\widetilde{H}}^{G(d,1,n)}(E^{(\lambda^{(0)},\emptyset,\emptyset,\ldots,\emptyset)} \boxtimes E^{(\emptyset,\lambda^{(1)},\emptyset,\ldots,\emptyset)} \boxtimes \cdots \boxtimes E^{(\emptyset,\emptyset,\emptyset,\ldots,\lambda^{(d-1)})}), \tag{4.3}$$

where $\widetilde{H} := \prod_{i=0}^{d-1} G(d,1,k_i)$, which is naturally a subgroup of $G(d,1,n)$. Now note that $(\mathbb{Z}/d\mathbb{Z})^n \subseteq \widetilde{H}$. So we have $G(d,1,n) = S_n\widetilde{H}$ and $S_n \cap \widetilde{H} = H$, where $H := \prod_{i=0}^{d-1} S_{k_i}$. Hence, Mackey's formula shows that

$$\mathrm{Res}_{S_n}^{G(d,1,n)}(E^\lambda) = \mathrm{Ind}_H^{S_n}(E^{\lambda^{(0)}} \boxtimes E^{\lambda^{(1)}} \boxtimes \cdots \boxtimes E^{\lambda^{(d-1)}}). \tag{4.4}$$

Applying (4.2) yields the following formula for the restriction of irreducible representations from $G(d,1,n)$ to S_n:

$$\mathrm{Res}_{S_n}^{G(d,1,n)}(E^\lambda) = \sum_{\nu^{(i)} \in \mathcal{P}(k_0+\cdots+k_i)} c_{\lambda^{(0)},\lambda^{(1)}}^{\nu^{(1)}} c_{\nu^{(1)},\lambda^{(2)}}^{\nu^{(2)}} \cdots c_{\nu^{(d-2)},\lambda^{(d-1)}}^{\nu^{(d-1)}} E^{\nu^{(d-1)}}. \tag{4.5}$$

Let us denote by c_λ^ν the coefficient of E^ν in the above formula, for all $\nu \in \mathcal{P}(n)$. Lemma 2 can be then generalised as follows:

Lemma 4. *Let $\lambda = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(d-1)}) \in \mathcal{P}(d,n)$. Set $\alpha := \sum_{i=0}^{d-1} \lambda_1^{(i)}$. Then*

(1) for all $\nu \in \mathcal{P}(n)$ with $\nu_1 > \alpha$, we have $c_\lambda^\nu = 0$;

(2) there exists $\nu \in \mathcal{P}(n)$ such that $\nu_1 = \alpha$ and $c_\lambda^\nu > 0$.

Proof. (1) We have $c_\lambda^\nu = c_{\lambda^{(0)},\lambda^{(1)}}^{\nu^{(1)}} c_{\nu^{(1)},\lambda^{(2)}}^{\nu^{(2)}} \cdots c_{\nu^{(d-2)},\lambda^{(d-1)}}^{\nu^{(d-1)}}$, where $\nu^{(d-1)} = \nu$. If $c_\lambda^\nu \neq 0$, then, by Lemma 2 (1), we must have

$$\nu_1^{(i)} \leqslant \nu_1^{(i-1)} + \lambda_1^{(i)} \quad \text{for all } i = 1, 2, \ldots, d-1$$

where we take $\nu^{(0)} := \lambda^{(0)}$. We deduce that

$$\nu_1^{(i)} \leqslant \lambda_1^{(0)} + \cdots + \lambda_1^{(i)} \quad \text{for all } \ i = 1, 2, \ldots, d-1.$$

In particular, for $i = d-1$, we obtain $\nu_1 \leqslant \alpha$.
(2) Set again $\nu^{(0)} := \lambda^{(0)}$. Following Lemma 2 (2), we can define inductively $\nu^{(i)} \in \mathcal{P}(|\nu^{(i-1)}|+|\lambda^{(i)}|)$, for all $i = 1, 2, \ldots, d-1$, such that

$$\nu_1^{(i)} = \nu_1^{(i-1)} + \lambda_1^{(i)} \quad \text{and} \quad c_{\nu^{(i-1)},\lambda^{(i)}}^{\nu^{(i)}} > 0.$$

We deduce that $\nu^{(i)} \in \mathcal{P}(|\lambda^{(0)}|+\cdots+|\lambda^{(i)}|)$ and that $\nu_1^{(i)} = \lambda_1^{(0)} + \cdots + \lambda_1^{(i)}$, for all $i = 1, 2, \ldots, d-1$.

Set $\nu := \nu^{(d-1)}$. We then have $\nu \in \mathcal{P}(n)$, $\nu_1 = \alpha$ and

$$c_\lambda^\nu = c_{\lambda^{(0)},\lambda^{(1)}}^{\nu^{(1)}} c_{\nu^{(1)},\lambda^{(2)}}^{\nu^{(2)}} \cdots c_{\nu^{(d-2)},\lambda^{(d-1)}}^{\nu^{(d-1)}} > 0.$$

□

4.2. **Determination of $\mathcal{R}(d,n)$.** In order to obtain a description of $\mathcal{R}(d,n)$, we will combine Corollary 1 with Lemma 4.

Proposition 2. *Let $\lambda = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(d-1)}) \in \mathcal{P}(d,n)$. The Young diagrams of all direct summands of $\mathrm{Res}^{G(d,1,n)}_{S_n}(E^\lambda)$ have at most two columns if and only if $\sum_{i=0}^{d-1} \lambda_1^{(i)} \leqslant 2$.*

Proof. Set $\alpha := \sum_{i=0}^{d-1} \lambda_1^{(i)}$. First suppose that $\alpha \leqslant 2$, and let E^ν be a direct summand of $\mathrm{Res}^{G(d,1,n)}_{S_n}(E^\lambda)$. By Lemma 4 (1), if $\nu_1 > 2 \geq \alpha$, then $c^\nu_\lambda = 0$. So we must have $\nu_1 \leqslant 2$.

On the other hand, if $\alpha > 2$, then, by Lemma 4 (2), there exists $\nu \in \mathcal{P}(n)$ such that $\nu_1 = \alpha > 2$ and $c^\nu_\lambda > 0$. Thus, E^ν is a direct summand of $\mathrm{Res}^{G(d,1,n)}_{S_n}(E^\lambda)$ whose Young diagram has more than two columns. $\square$

Now, Proposition 2 combined with Corollary 1 yields:

Corollary 3. *Let $n \geq 3$. We have*

$$\mathcal{R}(d,n) = \left\{ \lambda \in \mathcal{P}(d,n) \;\middle|\; \sum_{i=0}^{d-1} \lambda_1^{(i)} \leqslant 2 \right\}. \tag{4.6}$$

That is, $E^\lambda \in \mathrm{Irr}(\mathrm{YTL}_{d,n}(u))$ if and only if the Young d-diagram of λ has at most two columns in total.

4.3. **Pieri's rule for type $G(d,1,n)$.** The following result, known as "Pieri's rule", derives from the formulas in Subsection 4.1 with the use of Lemma 3.

Proposition 3. *Let $n = k + l$ where $k \geq 0$, $l \geq 1$. Let $\mu \in \mathcal{P}(d,k)$ and $\lambda \in \mathcal{P}(d,n)$. Then E^λ is a direct summand of $\mathrm{Ind}^{G(d,1,n)}_{G(d,1,k)\times S_l}(E^\mu \boxtimes E^{(l)})$ if and only if the Young d-diagram of λ can be obtained from that of μ by adding l boxes, with no two in the same column.*

For $d = 1$, the above result is the classical Pieri's rule for the symmetric group (cf. [GePf, 6.1.7]). For $d > 1$, the proof is similar to the one for type $B_n \cong G(2,1,n)$ (cf. [GePf, 6.1.9]).

The following corollary (case $l = 3$) gives an alternative proof of Corollary 3.

Corollary 4. *Let $n \geq 3$ and set $k := n - 3$. Let $\lambda \in \mathcal{P}(d,n)$. There exists $\mu \in \mathcal{P}(d,k)$ such that E^λ is a direct summand of $\mathrm{Ind}^{G(d,1,n)}_{G(d,1,k)\times S_3}(E^\mu \boxtimes E^{(3)})$ if and only if $\sum_{i=0}^{d-1} \lambda_1^{(i)} > 2$.*

5. Dimension of the Yokonuma–Temperley–Lieb algebra

Since the Yokonuma–Temperley–Lieb algebra $\mathrm{YTL}_n(u)$ is split semisimple over $\mathbb{C}(u)$, we must have

$$\dim_{\mathbb{C}(u)}(\mathbb{C}(u)\mathrm{YTL}_{d,n}(u)) = \sum_{\lambda\in\mathcal{R}(d,n)} (\dim(\varrho^\lambda))^2 = \sum_{\lambda\in\mathcal{R}(d,n)} (\dim(\rho^\lambda))^2 = \sum_{\lambda\in\mathcal{R}(d,n)} (\dim(E^\lambda))^2. \tag{5.1}$$

We have that $\dim(E^\lambda)$ is equal to the number of semistandard d-tableaux of shape λ and weight $(1,1,\ldots,1)$; we call these tableaux standard d-tableaux of shape λ.

Now, for $d = 1$, it is well-known that the dimension of the classical Temperley–Lieb algebra $\mathrm{TL}_n(u)$ is given by the *n-th Catalan number*

$$C_n := \frac{1}{n+1}\binom{2n}{n} = \frac{1}{n+1}\sum_{k=0}^{n}\binom{n}{k}^2. \tag{5.2}$$

We will see that the n-th Catalan appears also in the dimension formula of the Yokonuma–Temperley–Lieb algebra $\mathrm{YTL}_{d,n}(u)$ for $d > 1$.

Proposition 4. *We have*

$$\dim_{\mathbb{C}(u)}(\mathbb{C}(u)\mathrm{YTL}_{d,n}(u)) = \frac{d\,(nd-n+d+1)}{2}\,C_n - d\,(d-1). \tag{5.3}$$

Proof. Following the description of the d-partitions in $\mathcal{R}(d,n)$ by Corollary 3, we have that

$$\mathcal{R}(d,n) = \mathcal{R}_1(d,n) \sqcup \mathcal{R}_2(d,n),$$

where

$$\mathcal{R}_1(d,n) = \left\{ \lambda \in \mathcal{P}(d,n) \;\middle|\; \exists\, i \in \{0,1,\dots,d-1\} \text{ such that } \lambda^{(i)} \in \mathcal{R}(1,n) \text{ and } \lambda^{(j)} = \emptyset,\, \forall\, j \neq i \right\}$$

and

$$\mathcal{R}_2(d,n) = \left\{ \lambda \in \mathcal{P}(d,n) \;\middle|\; \exists\, i_1 \neq i_2 \in \{0,1,\dots,d-1\} \text{ such that } \lambda_1^{(i_1)} = \lambda_1^{(i_2)} = 1 \text{ and } \lambda^{(j)} = \emptyset,\, \forall\, j \neq i_1, i_2 \right\}.$$

We have

$$\sum_{\lambda \in \mathcal{R}_1(d,n)} (\dim(E^\lambda))^2 = d \dim_{\mathbb{C}(u)}(\mathrm{TL}_n(u)) = d\,C_n.$$

So it remains to calculate

$$\sum_{\lambda \in \mathcal{R}_2(d,n)} (\dim(E^\lambda))^2.$$

Let $\lambda \in \mathcal{R}_2(d,n)$. Then there exist $i_1 \neq i_2 \in \{0,1,\dots,d-1\}$ such that $\lambda_1^{(i_1)} = \lambda_1^{(i_2)} = 1$ and $\lambda^{(j)} = \emptyset$, for all $j \neq i_1, i_2$. Assume that $\lambda^{(i_1)} = (1,1,\dots,1)$ has k parts, for some $k \in \{1,2,\dots,n-1\}$. Then $\lambda^{(i_2)} = (1,1,\dots,1)$ has $n-k$ parts, and

$$\dim(E^\lambda) = \binom{n}{k}.$$

Since there are $\binom{d}{2}$ choices for the pair (i_1, i_2), we conclude that

$$\sum_{\lambda \in \mathcal{R}_2(d,n)} (\dim(E^\lambda))^2 = \binom{d}{2} \sum_{k=1}^{n-1} \binom{n}{k}^2.$$

Thus, we have

$$\dim_{\mathbb{C}(u)}(\mathbb{C}(u)\mathrm{YTL}_{d,n}(u)) = d\,C_n + \frac{d(d-1)}{2} \sum_{k=1}^{n-1} \binom{n}{k}^2.$$

Due to (5.2), we can replace the sum in the above formula by $(n+1)\,C_n - 2$; this yields (5.3). $\square$

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